HAMILTONIANS ON DISCRETE STRUCTURES: JUMPS OF THE INTEGRATED DENSITY OF STATES AND UNIFORM CONVERGENCE

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ABSTRACT. We study equivariant families of discrete Hamiltonians on amenable geometries and their integrated density of states (IDS). We prove that the eigenspace of a fixed energy is spanned by eigenfunctions with compact support. The size of a jump of the IDS is consequently given by the equivariant dimension of the subspace spanned by such eigenfunctions. From this we deduce uniform convergence (w.r.t. the spectral parameter) of the finite volume approximants of the IDS. Our framework includes quasiperiodic operators on Delone sets, periodic and random operators on quasi-transitive graphs, and operators on percolation graphs.

1. INTRODUCTION

The integrated density of states, in the following abbreviated as IDS, can be defined for a variety of models, ranging from Hamiltonians in the quantum theory of solids to Laplacians on p-cochains on CW-complexes. We refer e.g. to [39, 42, 25, 18, 2, 3, 34, 46, 19] which represent but a fraction of the literature devoted to this topic. While some of these references concern operators on continuum configuration space, in the present paper we restrict ourselves to models on discrete spaces. Under certain geometric conditions the IDS can be approximated by its analogues associated to finite volume restrictions of the Hamiltonian. Here the approximation is a priori understood in the sense of weak convergence of measures. We show that under an amenability condition the following properties are universal for a wide range of models:

(a) If λ is a point of discontinuity of the IDS, there exist compactly supported eigenfunctions to λ, and these compactly supported eigenfunctions actually span the whole eigenspace of λ.
(b) The IDS can be approximated by its finite volume analogues *uniformly* in the spectral variable, i.e. with respect to the supremum norm.
(c) The size of each jump of the IDS can be approximated by the jumps of the finite volume analogues.

Some of our models are random, i.e. concern a whole family of operators. For such models the three properties above hold *almost surely*. We present our results in a general setting. They can be applied to a variety of models considered before, for instance, Anderson and quantum percolation models [40, 5, 4, 45, 20, 1], quasi-crystal Hamiltonians on Delone sets [15, 16, 29], Laplacians on p-cochains on complexes [7, 8, 10], Harper operators [41, 36, 35], random hopping models [22], and Hamiltonians associated to percolation on tilings [17, 37].

The geometric framework which allows us to treat all these models at once is given by an action of an amenable group Γ on a metric space X such that the two are roughly isometric. It is not surprising that the notion of rough isometry is fitting in this context since it has been proposed in [14] as a tool for studying geometric properties of a space 'at infinity'. On the other hand, it is well known that the IDS does not change under compactly supported perturbations. Due to the chosen setting we can treat in parallel situations where the underlying group is continuous and discrete.

There is no model known to us where all of the features (a) – (c) were obtained before. Partial results, however, were known for some specific models mentioned above. Let us now briefly discuss these earlier partial results recovered in our framework. Results about pointwise convergence were

obtained in, e.g., [15, 39, 36, 35]. Note that in the literature special attention was devoted to convergence at the bottom of the spectrum [7, 8, 10], since it corresponds to approximation of Betti-numbers. Results on uniform convergence of the IDS for models with finite local complexity were obtained in [32, 27, 11]. Statement (a) was established for periodic models related to abelian groups in [24, 23]. Weaker statements about the characterisation of the jumps of the IDS were subsequently proven in [21] and [45]. Approximation of the jumps of the IDS by the finite volume analogues are contained in [35]. Let us stress that jumps of the IDS actually do occur for several models — like quasi-periodic models [21] and percolation Hamiltonians [4, 45]. Thus uniform convergence is by no means automatically implied by pointwise convergence.

Our theorems show that uniform convergence of the IDS is an universal phenomenon and does not depend on specific features of the model, like number-theoretic properties or finite local complexity. We hope that this clarification will be helpful for the study of finer properties of the IDS, which may be indeed model-dependent. Among those are: the quantisation of jumps of the IDS [9, 43], their location [36, 6, 45], and the low energy behaviour of the IDS. Note that this low energy behaviour has been a recent focus of attention in particular for percolation models (see e.g. [20, 38, 1]) while surveys of results for other models can be e.g. found in [34, 19].

In the context of uniform convergence, we would also like to emphasize that our method of proof works without a uniform ergodic theorem. This is a fundamental difference to the earlier considerations of [32, 27] (see [26] as well). On the other hand the question whether such an ergodic theorem holds in the present contexts remains open. We consider this an interesting question.

This paper is organized as follows: In Section 2 we present our results and fix the notation. Section 3 provides the necessary background information on rough isometries. The next four sections are devoted to the proofs of our three main results. Section 8 discusses how results on aperiodic order fit into our framework. In Section 9 we show how earlier results on periodic operators on graphs can be recovered by our approach. Section 10 discusses percolation models and finally, Section 11 is devoted to models related to percolation on Delone sets.

2. Setting and results

Let (X, d) be a locally compact metric space with a countable basis of the topology. Let Γ be a locally compact amenable unimodular group with an invariant metric d_Γ such that every ball is precompact. The invariant Haar measure of a Borel mesurable set $A \subset \Gamma$ is denoted by $|A|$. Let Γ act continuously by isometries on X such that the following two properties hold:

- There exists a fundamental domain F' with compact closure F, which is a countable union of compact sets.
- The map $\Phi \colon X \longrightarrow \Gamma, x \mapsto \gamma$, whenever $x \in \gamma F'$, is a *rough isometry* i.e. there exist $b \geq 0$ and $a \geq 1$ with

$$\frac{1}{a} d_\Gamma(\Phi(x), \Phi(y)) - b \leq d(x, y) \leq a d_\Gamma(\Phi(x), \Phi(y)) + b$$

for all $x, y \in X$.

Note that all these assumptions are automatically satisfied if both Γ and X are discrete and Γ acts cocompactly and freely on X. Models in such a geometric setting are considered in Sections 9 and 10 below.

Our operators will be families of operators indexed by elements of a certain topological space. This topological space will be considered next (see Appendix for details). We consider the following family of uniformly discrete subsets of X

$$\mathcal{D} := \{A \subset X \mid d(x, y) \geq 1, \ \text{for} \ x, y \in A \ \text{with} \ x \neq y\}.$$

The lower bound $d(x, y) \geq 1$ could be replaced by any other positive number. Whenever Y is a compact space we can then equip the set of functions $f : A \longrightarrow Y$, where $A \in \mathcal{D}$, with the vague topology and obtain a compact space $\mathcal{D}_Y$. In fact, there is a choice of Y which is in some sense universal, and which we discuss in the appendix. There we show in particular that the space

$$\tilde{\mathcal{D}} := \{(A, h) \mid A \in \mathcal{D}, h \colon A \times A \to \mathbb{C}^*\}.$$

is naturally equipped with the vague topolgy and moreover compact. Here $\mathbb{C}^*$ is an arbitrary compactification of $\mathbb{C}$. The mapping $(A, h) \mapsto \sum_{x,y \in A} g(x, y)h(x, y) \in \mathbb{C}$ is continuous for any continuous $g \colon X \times X \to \mathbb{C}$ with compact support. In particular, it is measurable for any continuous $g \colon X \times X \to \mathbb{C}$. This measurablility then holds for as well for any nonnegative measurable $g \colon X \times X \to \mathbb{C}$.

The action of Γ on X induces an action on $\tilde{\mathcal{D}}$ by $\gamma(A, h) := (\gamma A, \gamma h)$ where $\gamma A := \{\gamma a \mid a \in A\}$ and $(\gamma h)(x, y) = h(\gamma^{-1} x, \gamma^{-1} y)$ for $x, y \in \gamma A$. Let μ be an Γ-invariant probability measure on $\tilde{\mathcal{D}}$, whose topological support we denote by Ω. Then, Ω is a closed and hence compact subset of $\tilde{\mathcal{D}}$. For $\omega = (A, h) \in \tilde{\mathcal{D}}$ we use the notation $X(\omega) := A$ for the projection on the first component. The latter is a discrete metric subspace of X and gives rise to $\ell^2(X(\omega))$ with the natural counting measure $\delta_{X(\omega)} := \sum_{x \in X(\omega)} \delta_x$. For $\omega \in \tilde{\mathcal{D}}$ and $\Lambda \subset X$ we denote by $\Lambda(\omega)$ the intersection $\Lambda \cap X(\omega)$ and by $\omega(\Lambda)$ the cardinality $|\Lambda(\omega)| = \delta_{X(\omega)}(\Lambda)$.

The measure μ induces a Γ-invariant measure m on X by

$$m(\Lambda) := \int_\Omega \omega(\Lambda)d\mu(\omega).$$

For any $S < \infty$, let M_S be the maximal number of points with mutual distance at least 1 contained in a ball of radius S in X. Then M_S is finite for all S by the very definition of $\mathcal{D}$. Thus, the measure m is bounded by M_S on a ball of size S and hence m is finite on bounded sets. We assume that Ω does not consist of the empty set only. This implies $m(X) \neq 0$, since the vague topology on Ω is Hausdorff. As F' is a countable union of compact sets, the set IF' is measurable for any measurable $I \subset \Gamma$ by standard monotone class arguments. The map $\Gamma \supset I \mapsto m(IF')$ gives an invariant measure on Γ. By uniqueness of the Haar measure we infer that $\mathrm{dens}(m) := \frac{m(IF')}{|I|}$ is a nonnegative constant and hence

$$m(IF') = \mathrm{dens}(m)|I|$$

for all $I \subset \Gamma$ compact with $|I| \neq 0$. Since $m(X) > 0$, $\mathrm{dens}(m)$ is actually strictly positive. A direct calculation using unimodularity now shows that $u := \frac{1}{|I|}\chi_{IF'}$ satisfies

(1) $\qquad\qquad u$ has compact support and $1 = \int u(\gamma^{-1}x)d\gamma$ for all $x \in X$

for any $I \subset \Gamma$ compact with $|I| > 0$ (see Proposition 4.1).

Denote by $\mathcal{H}$ the direct integral Hilbert space $\int_\Omega^\oplus d\mu(\omega)\,\ell^2(X(\omega))$. Each $\omega = (X(\omega), h) \in \tilde{\mathcal{D}}$ gives rise to an operator

$$H_\omega \colon c_0(X(\omega)) \to \ell^2(X(\omega)), \quad (H_\omega v)(x) := \sum_{y \in X(\omega)} H_\omega(x, y)v(y), \quad \text{with } H_\omega(x, y) := h(x, y)$$

where $c_0(X(\omega))$ denotes the complex valued functions with compact support in $X(\omega)$. If $H_\omega \colon \ell^2(X(\omega)) \to \ell^2(X(\omega))$ is bounded by $C \in \mathbb{R}$ for all $\omega \in \Omega$, we call

$$H \colon \mathcal{H} \to \mathcal{H}, \quad H = \int_\Omega^\oplus d\mu(\omega)\, H_\omega$$

a *bounded decomposable operator*. Note that in this case the values of $|H_\omega(x, y)|$ are bounded by C for all $\omega \in \Omega$ and $x, y \in X(\omega)$. A decomposable operator is called of *finite hopping range* if there exists an $R < \infty$ such that for all $\omega \in \Omega$ and $x, y \in X(\omega)$, $d(x, y) \geq R$ implies $H_\omega(x, y) = 0$.

Now, let for every $\gamma \in \Gamma$ a function $s_\gamma \colon X \longrightarrow \{z \in \mathbb{C} : |z| = 1\}$ be given. A decomposable operator is called *equivariant* (w.r.t. the family s_γ) if and only if

$$s_\gamma(x)H_{\gamma\omega}(\gamma x, \gamma y)\overline{s_\gamma(y)} = H_\omega(x, y)$$

for all $\gamma \in \Gamma$, $\omega \in \Omega$, and $x, y \in X(\omega)$. Of course, this is equivalent to

$$HT_\gamma = T_\gamma H,$$

for all $\gamma \in \Gamma$ with the unitary map $T_\gamma \colon \mathcal{H} \longrightarrow \mathcal{H}$, $(T_\gamma f)_\omega(x) = s_{\gamma^{-1}}(x)f_{\gamma^{-1}\omega}(\gamma^{-1}x)$.

Remark. It turns out that for the results and proofs of this paper it does not matter whether s_γ is a non-trivial function or $s_\gamma \equiv 1$. The reason is that all calculations concern the diagonal matrix elements either of one single operator H or of a product of two operators HK. In this case, the terms $s_\gamma(x)$ and $\overline{s_\gamma(x)}$ cancel and equivariance of operators H and K gives

$$H_{\gamma\omega}(\gamma x, \gamma x) = H_\omega(x, x)$$

and

$$(HK)_{\gamma\omega}(\gamma x, \gamma x) = \sum_{y \in X(\omega)} H_\omega(x, y)K_\omega(y, x).$$

Thus the function s_γ can be immediately eliminated.

We are interested in operators H satisfying (A) given as follows:

(A) $H \colon \mathcal{H} \to \mathcal{H}$ is a bounded, selfadjoint, decomposable, equivariant operator H of finite hopping range.

As usual we denote the spectral family of a selfadjoint operator T by E_T.

Theorem 2.1. *Let H satisfy (A). Let u satisfy (1). Then, there exist a unique measure ν_H on $\mathbb{R}$ with*

(2) $$\nu_H(\varphi) = \frac{1}{\mathrm{dens}(m)} \int_\Omega \mathrm{Tr}(u\varphi(H_\omega))d\mu(\omega)$$

for all $\varphi \in C_c(\mathbb{R})$. The measure ν_H does not depend on u provided (1) is satisfied. It is a spectral measure for H, i.e. $\nu_H(B) = 0$ if and only if $E_H(B) = 0$.

The distribution function of ν_H is denoted by N_H, i.e.

$$N_H \colon \mathbb{R} \longrightarrow \mathbb{R}, \ N_H(\lambda) := \nu_H((-\infty, \lambda]),$$

and called the *integrated density of states*, IDS for short.

Theorem 2.2. *Let H satisfy (A). Let $\lambda \in \mathbb{R}$ be arbitrary. Let $P_{comp} := (P_{comp,\omega})$, where $P_{comp,\omega}$ is the projection onto the subspace of $\ell^2(X(\omega))$ spanned by compactly supported solutions of $(H_\omega - \lambda)u = 0$. Then,*

(3) $$E_H(\{\lambda\}) = P_{comp}.$$

The theorem does not assume ergodicity. If ergodicity holds, then the statement can be strengthened to give the following corollary.

Corollary 2.3. *Let H satisfy (A). Let μ be ergodic. Let $\lambda \in \mathbb{R}$ be arbitrary. Then, the following assertions are equivalent.*

(i) *N_H is discontinuous at λ.*
(ii) *For a set of positive measure $\Omega' \subset \Omega$ there exists a compactly supported nontrivial solution $u \in \ell^2(X(\omega))$ of $(H_\omega - \lambda)u = 0$ for each $\omega \in \Omega'$.*
(iii) *For almost every $\omega \in \Omega$ the space of ℓ^2 solutions u to $(H_\omega - \lambda)u = 0$ is spanned by compactly supported nontrivial solutions.*

Since Γ is by assumption amenable, there exists a Følner sequence in Γ. This is by definition a sequence of compact, nonempty sets $(I_n)_n, I_n \subset \Gamma$ such that for any compact $K \subset \Gamma$ and $\epsilon > 0$ we have $|I_n \triangle KI_n| < \epsilon|I_n|$ if n is large enough. By passing to a subsequence one may assume that $(I_n)_n$ is tempered, i.e. that for some $C \in (0, \infty)$ and all $n \in \mathbb{N}$ the condition $|\bigcup_{k<n} I_k^{-1} I_n| \leq C|I_n|$ holds.

Let (I_n) be a tempered Følner sequence in Γ and define $\Lambda_n := I_n F$. Let $H_{\omega,n}$ be the restriction of H_ω to $\ell^2(\Lambda_n(\omega))$ i.e.

$$H_{\omega,n} = p_{\Lambda_n(\omega)} H_\omega i_{\Lambda_n(\omega)}$$

with the natural projection $p_{\Lambda_n(\omega)} \colon \ell^2(X(\omega)) \longrightarrow \ell^2(\Lambda_n(\omega))$ and the natural injection $i_{\Lambda_n(\omega)} \colon \ell^2(\Lambda_n(\omega)) \longrightarrow \ell^2(X(\omega))$. Note that the space $\ell^2(\Lambda_n(\omega))$ is finite dimensional. Let $N_{\omega,n}$ be the corresponding eigenvalue counting function, i.e.

$$N_{\omega,n}(\lambda) := \mathrm{Tr}\, E_{H_{\omega,n}}((-\infty, \lambda]) = \dim \mathrm{ran}\, E_{H_{\omega,n}}((-\infty, \lambda]).$$

Let $\nu_{\omega,n}$ be the measure whose distribution function is $N_{\omega,n}$.

Theorem 2.4. *Let H satisfy (A). Assume that μ is ergodic. Then, the sequence $\frac{N_{\omega,n}}{\omega(\Lambda_n)}$ converges with respect to the supremum norm $\|\cdot\|_\infty$ to N_H for μ-almost every $\omega \in \Omega$,*

Even without the assumption of ergodicity we obtain uniform convergence of the distribution functions $\frac{N_{\omega,n}}{\omega(\Lambda_n)}$, but the limit cannot be described explicitly, see Remark 7.4 for details.

3. Rough isometry and linear algebra

In this section, we collect some basic facts about rough isometries and subspace dimensions. They provide our working tools for the subsequent sections.

Let the assumptions of the previous section hold. Let $p \in X$ be fixed with $\Phi(p) = \mathrm{id} \in \Gamma$. For $\Lambda \subset X$, $x \in X$ and $r > 0$ we define $d(x,\Lambda) := \inf\{d(x,y) \mid y \in \Lambda\}$ and

$$\Lambda^r := \{x \in X \mid d(x,\Lambda) < r\}, \ \Lambda_r := \{x \in X : d(x, X \setminus \Lambda) > r\}, \ \partial^r \Lambda := \Lambda^r \setminus \Lambda_r.$$

For $I \subset \Gamma$ we use the analogous notation with d replaced by d_Γ. Denote by $B_s(\gamma)$ the open ball around $\gamma \in \Gamma$ with radius s. If γ is the identity we simply write B_s for $B_s(\gamma)$.

Lemma 3.1. *For $r \geq 0$ and any $s \geq ar + b$, the inclusion $F^r \subset B_s F'$ holds.*

Proof. For any $x \in F^r$ there exists $y \in F'$ with $d(x,y) < r$ and a unique $\gamma \in \Gamma$ such that $x \in \gamma F'$. Thus $d_\Gamma(\gamma, \mathrm{id}) \leq a d(x,y) + b < ar + b$, implying $\gamma \in B_s$. $\qquad \square$

Proposition 3.2. *For each $r > 0$, let $q \geq ar + ab + b$ and $s \geq ar + b$ be given. Then for all $I \subset \Gamma$:*

$$(IF)^r \subset I^s F' \ \text{ and } \ (I_q F) \subset (IF')_r \ \text{ and } \ \partial^r(IF) \subset (\partial^q I)F'.$$

Proof. Since Γ acts on X by isometries, $(\gamma F)^r = \gamma F^r$. By the previous Lemma $(IF)^r = IF^r \subset IB_s F' = I^s F'$ and the first inclusion holds. For $x \in I_q F'$ there exist unique $\gamma \in I_q, x_0 \in F'$ such that $x = \gamma x_0$. By definition $d_\Gamma(\gamma, \beta) > q$ for all $\beta \in \Gamma \setminus I$. Let $y \in X \setminus IF'$ be arbitrary. Then there are unique $y_0 \in F', \alpha \in \Gamma \setminus I$ with $y = \alpha y_0$. Consequently $d(x,y) = d(\gamma x_0, \alpha y_0) \geq d_\Gamma(\gamma, \alpha)/a - b > q/a - b$ and thus $x \in (IF')_r$ for $r = q/a - b$. Since $I_{ar+ab+b}F \subset I_{ar+ab+b}B_b F' \subset I_{ar+ab}F' \subset (IF')_r$ we proved the second inclusion. The last inclusion is a combination of the previous two inclusions. $\qquad \square$

Recall that M_S denotes the maximal number of points with distance at least one contained in a ball of radius S in X.

Proposition 3.3. *For any $\rho > 0$ and $C := \dfrac{M_{2a(\rho+b)}}{|B_\rho|}$*

$$\omega(IF') \leq C \, |I^\rho|,$$

for all $I \subset \Gamma$ and all $\omega \in \mathcal{D}$.

Recall that for I precompact $|I^\rho|$ is finite.

Proof. Choose $S > ab$ and define $N_X(S, A)$ to be the maximal number of points in $A \subset X$ with distance at least S between them and $N_\Gamma(S, I)$ to be the maximal number of points in $I \subset \Gamma$ with distance at least S between them. Then,

$$\omega(IF') \leq N_X(1, IF') \leq M_{2S} \, N_X(S, IF') \leq M_{2S} \, N_\Gamma(S/a - b, I).$$

Set $\rho := S/a - b > 0$. Since $|B_\rho| N_\Gamma(\rho, I) = \big| \bigcup_{i=1}^{N_\Gamma(\rho, I)} B_\rho(\gamma_i) \big| \leq |I^\rho|$ we obtain the claim. $\qquad \square$

The proposition has the following consequence, which is crucial for our results.

Proposition 3.4. *Let (I_n) be a Følner sequence. Then for arbitrary $r \geq 0$ and $\omega \in \mathcal{D}$,*

$$\lim_{n \to \infty} \frac{\omega((\partial^r I_n)F')}{|I_n|} = 0 \ \text{ and } \ \lim_{n \to \infty} \frac{\omega(\partial^r \Lambda_n)}{|I_n|} = 0.$$

Proof. Proposition 3.3 gives us $\omega((\partial^r I)F') \leq \frac{M_{a\rho+b}}{|B_\rho|}|(\partial^r I)^\rho| \leq \frac{M_{a\rho+b}}{|B_\rho|}|(\partial^{r+\rho}I)|$, thus the Følner property of (I_n) implies the first equality. Since $r > 0$ was arbitrary and $\partial^r(IF) \subset (\partial^q I)F$, for $q \geq ar + ab + b$, the second equality follows immediately. $\qquad\square$

Lemma 3.5. *For $\omega \in \mathcal{D}$ and $\Lambda \subset X$ let U be a subspace of $\ell^2(\Lambda(\omega))$. For $S \geq 0$ denote by U_S the subspace consisting of all functions in U which vanish outside of Λ_S. Then,*

$$0 \leq \dim(U) - \dim(U_S) \leq \omega(\partial^S\Lambda).$$

Proof. Let $Q\colon U \longrightarrow \ell^2\big((\Lambda \setminus \Lambda_S)(\omega)\big)$ be the natural restriction map. Then,

$$\dim(U) - \dim(\ker Q) = \dim(\operatorname{ran} Q).$$

As $\ker Q = U_S$ and $\dim(\operatorname{ran} Q) \leq \omega(\Lambda \setminus \Lambda_S) \leq \omega(\partial^S\Lambda)$, the statement follows. $\qquad\square$

4. Covariant operators and their trace

In this section we briefly discuss covariant operators and their natural trace. We will then provide a proof of Theorem 2.1.

Let $\mathcal{N}$ be the set of all covariant decomposable bounded operators. Obviously, $\mathcal{N}$ is a vector space in the natural way. Moreover, by $T^* := (T_\omega^*)$ and $TS := (T_\omega S_\omega)$ it becomes a $*$-algebra. In fact, it is even a von Neumann algebra. We will not use this in the sequel. We will use that for any selfadjoint $T = (T_\omega) \in \mathcal{N}$ the operator

$$f(T) = (f(T_\omega)) \text{ for a bounded and measurable } f\colon \mathbb{R} \to \mathbb{C}$$

defined by spectral calculus is an element of $\mathcal{N}$ as well. We start by having a look at functions which satisfy (1).

Proposition 4.1. *For every measurable precompact $I \subset \Gamma$ with $0 < |I|$, the function $|I|^{-1}\chi_{IF'}$ satisfies (1).*

Proof. As I is precompact and the action of Γ is continuous by assumption, the function $\chi_{IF'}$ has compact support. Let $x \in X$ be arbitrary. Then, x can be uniquely written as $x = \alpha x_0$ with $\alpha \in \Gamma$ and $x_0 \in F'$. A short calculation then shows $\chi_{IF'}(\gamma^{-1}x) = \chi_{\alpha I^{-1}}(\gamma)$ and the claim follows easily from unimodularity. $\qquad\square$

For a function u on X and $I \subset \Gamma$ we define u_I by $u_I(x) := \int_I u(\gamma^{-1}x)d\gamma$.

Proposition 4.2. *If u satisfies (1), so does $|I|^{-1}u_I$ for any precompact measurable $I \subset \Gamma$ with $0 < |I|$.*

Proof. This follows by a direct calculation using Fubini theorem and unimodularity. $\qquad\square$

Theorem 4.3. *Let u satisfy (1). Then, the map*

$$\tau\colon \mathcal{N} \longrightarrow \mathbb{R}, \ \tau(T) := \frac{1}{\operatorname{dens}(m)} \int \operatorname{Tr}(uT_\omega)d\mu(\omega)$$

does not depend on u and satisfies the following properties:
- *τ is faithful, i.e. for $T \geq 0$, $\tau(T) = 0$ if and only if $T = 0$.*
- *τ has the trace property, i.e. $\tau(ST) = \tau(TS)$ for all $T, S \in \mathcal{N}$.*

Proof. This can essentially be obtained from the groupoid theoretical considerations in [28]. For the convenience of the reader we give a direct proof (see [31, 30] for similar calculations as well).

We first show that τ does not depend on u. Let u and v satisfying (1) be given. Then,

$$
\begin{aligned}
\int_\Omega \sum_{x \in X(\omega)} u(x) T_\omega(x, x) d\mu(\omega) &= \int_\Omega \sum_{x \in X(\omega)} u(x) T_\omega(x, x) \left(\int_\Gamma v(\gamma^{-1} x) d\gamma \right) d\mu(\omega) \\
\text{(Fubini)} &= \int_\Omega \int_\Gamma \left(\sum_{x \in X(\omega)} u(x) T_\omega(x, x) v(\gamma^{-1} x) \right) d\gamma d\mu(\omega) \\
\text{(covariance)} &= \int_\Omega \int_\Gamma \left(\sum_{y \in \gamma^{-1}\omega} u(\gamma y) T_{\gamma^{-1}\omega}(y, y) v(y) \right) d\gamma d\mu(\omega) \\
\text{(μ invariant)} &= \int_\Omega \int_\Gamma \sum_{y \in X(\omega)} u(\gamma y) T_\omega(y, y) v(y) d\gamma d\mu(\omega) \\
\text{(Fubini)} &= \int_\Omega \sum_{y \in X(\omega)} \left(\int_\Gamma u(\gamma y) d\gamma \right) T_\omega(y, y) v(y) d\mu(\omega) \\
&= \int_\Omega \sum_{y \in X(\omega)} v(y) T_\omega(y, y) d\mu(\omega).
\end{aligned}
$$

and independence is proven. By independence of u and Proposition 4.1 we can replace u by $\frac{1}{|I|}\chi_{IF'}$ for any precompact $I \subset \Gamma$. If $T \geq 0$ and $\tau(T) = 0$ we have $\chi_{B_r F'} T_\omega \chi_{B_r F'} = 0$ for almost all ω and any $r > 0$. Since the sequence $B_r F'$ exhausts X, T must be zero. Thus faithfulness is proven. We finally show $\tau(TK) = \tau(KT)$. The calculation is similar to the one to show independence of u. The definition of τ gives after inserting $1 = \int_\Gamma u(\gamma^{-1}y) d\gamma$ and using Fubini

$$
\begin{aligned}
\tau(TK) &= \int_\Omega \int_\Gamma \left(\sum_{x,y \in X(\omega)} T_\omega(x, y) K_\omega(y, x) u(x) u(\gamma^{-1}y) \right) d\gamma d\mu(\omega) \\
\text{(covariance)} &= \int_\Omega \int_\Gamma \left(\sum_{x,y \in \gamma^{-1}\omega} T_{\gamma^{-1}\omega}(x, y) K_{\gamma^{-1}\omega}(y, x) u(\gamma x) u(y) \right) d\gamma d\mu(\omega) \\
\text{(μ invariant)} &= \int_\Omega \int_\Gamma \left(\sum_{x,y \in X(\omega)} T_\omega(x, y) K_\omega(y, x) u(\gamma x) u(y) \right) d\gamma d\mu(\omega) \\
&= \tau(KT).
\end{aligned}
$$

This finishes the proof. $\qquad\square$

Definition 4.4. Let U be a subspace of $\mathcal{H} = \int^\oplus \ell^2(X(\omega)) d\mu(\omega)$ such that the orthogonal projection $P = P_U$ onto U belongs to $\mathcal{N}$. Then, $\tau(P)$ is called the equivariant dimension of U.

For later use we note the following consequence of Theorem 4.3 and Proposition 4.1.

Corollary 4.5. *For each measurable $I \subset \Gamma$ with $0 < |I| < \infty$ the equation*

$$
\tau(T) = \frac{1}{|I|} \frac{1}{\operatorname{dens}(m)} \int \operatorname{Tr}(\chi_{IF'} T_\omega) d\mu(\omega)
$$

holds for every $T \in \mathcal{N}$.

Proof. Set $J_0 := B_1 \cap I$ and $J_n := (B_{n+1} \setminus B_n) \cap I$ for $n \in \mathbb{N}$. Assume without loss of generality that $|J_n| > 0$ for all n. Then, I is the disjoint union of the J_n, $n \in \mathbb{N}_0$ and each J_n satisfies the assumption of Proposition 4.1. Hence, we obtain

$$
\frac{1}{|I|} \int \operatorname{Tr}(\chi_{IF'} T_\omega) d\mu(\omega) = \frac{1}{|I|} \sum_{n \in \mathbb{N}} \int \operatorname{Tr}(\chi_{J_n F'} T_\omega) d\mu(\omega) = \frac{1}{|I|} \sum_{n \in \mathbb{N}} |J_n| \operatorname{dens}(m) \tau(T) = \operatorname{dens}(m) \tau(T).
$$

$\qquad\square$

Proof of Theorem 2.1. Set $\nu_H(\varphi) = \tau(\varphi(H))$. Then ν_H is a positive functional and hence defines a unique measure. By faithfulness of τ it is a spectral measure. By definition, (2) holds.   □

5. EQUIVARIANT DIMENSION OF SUBSPACES AND JUMPS OF THE IDS

This section is devoted to a proof of Theorem 2.2. Note that we do not need ergodicity to derive this result.

Recall that every Følner sequence (I_n) in Γ induces a sequence $\Lambda_n := I_n F'$ of measurable subsets of X. By Proposition 3.4 (Λ_n) is a van Hove sequence in X. We start with a technical result.

Note that for $A_1 \subset A_2 \subset X$ we can canonically regard elements of $\ell^2(A_1(\omega))$ as elements of $\ell^2(A_2(\omega))$, as well, by extending them by zero.

Lemma 5.1. *Let $P \in \mathcal{N}$ with $P \geq 0$ and $\tau(P) > 0$ be given. Let $R > 0$ and a Følner sequence (I_n) in Γ be given. Then, there exists an $N \in \mathbb{N}$ and a set $\widetilde{\Omega}$ in Ω of positive measure such that for all $\omega \in \widetilde{\Omega}$*

$$\mathrm{ran}(p_{\Lambda_N(\omega)}P_\omega) \cap \ell^2(\Lambda_{N,R}(\omega)) \neq \{0\}.$$

Proof. Without loss of generality we can assume that the density $\mathrm{dens}(m)$ equals 1. As P be belongs to $\mathcal{N}$, the function $\omega \mapsto \|P_\omega\|$ is essentially bounded. We can assume without loss of generality that this constant is equal to 1. We set $\delta := \tau(P) > 0$. By Proposition 3.4, there exists $N \in \mathbb{N}$ with

$$\omega(\partial^R \Lambda_N) \leq \frac{\delta}{2}|I_N|$$

for all $\omega \in \Omega$. As by Corollary 4.5,

$$\delta = \tau(P) = \int \frac{1}{|I_N|} \mathrm{Tr}(\chi_{\Lambda_N} P_\omega)d\mu(\omega),$$

there exists a set $\widetilde{\Omega}$ of positive measure with

$$\frac{1}{|I_N|} \mathrm{Tr}(\chi_{\Lambda_N} P_\omega) \geq \delta$$

for all $\omega \in \widetilde{\Omega}$. This gives

$$\dim(\mathrm{ran}(p_{\Lambda_N(\omega)}P_\omega)) \geq \mathrm{Tr}(p_{\Lambda_N(\omega)}Pi_{\Lambda_N(\omega)}) \geq \delta|I_N|$$

for all $\omega \in \widetilde{\Omega}$. The statement follows from Lemma 3.5 with $U = \mathrm{ran}(p_{\Lambda_N(\omega)}P_\omega)$, since then $U_R = \mathrm{ran}(p_{\Lambda_N(\omega)}P_\omega) \cap \ell^2(\Lambda_{N,R}(\omega))$.   □

Lemma 5.2. *Let H satisfy (A). Let $\lambda \in \mathbb{R}$ be arbitrary. Set $P := E_H(\{\lambda\})$ and denote by P_{comp} the projection on the closure of the linear hull of compactly supported eigenfunctions to λ. Then, the following holds:*

  (a) *$P = P_{comp}$.*
  (b) *If $P_{comp} \neq 0$, then $\tau(P) > 0$.*

Proof. (a) If $P = 0$ the statement is clear. We therefore only consider the case $P \neq 0$, i.e. $\tau(P) > 0$. Assume $P \neq P_{comp}$. Then, $Q := P - P_{comp}$ is a projection with $Q \neq 0$. Hence $\tau(Q) > 0$ as τ is faithful. Set R to be twice the hopping range of H. By the previous lemma, there exist $N \in \mathbb{N}$ and a set $\widetilde{\Omega}$ of positive measure in Ω such that for each $\omega \in \widetilde{\Omega}$

$$\mathrm{ran}(p_{\Lambda_N(\omega)}Q_\omega) \cap \ell^2(\Lambda_{N,R}(\omega) \neq \{0\}.$$

By definition of the hopping range this gives compactly supported eigenfunctions in the range of Q_ω for all $\omega \in \widetilde{\Omega}$. This is a contradiction.

  (b) This follows as τ is faithful.   □

Proof of Theorem 2.2. This is a direct consequence of the previous lemma.   □

Proof of Corollary 2.3. As in Theorem 2.2, let $P_{comp,\omega}$ be the projection onto the space of compactly supported solutions of $(H_\omega - \lambda)u = 0$ and $P_{comp} = (P_{comp,\omega})$. Let Ω' be the set of $\omega \in \Omega$ with $P_{comp,\omega} \neq 0$. Then, Ω' is invariant and measurable, hence by ergodicity it has measure 0 or 1.

Now, by Theorem 2.2 and the faithfulness of τ, the function N_H is discontinuous at λ (i.e. (i) holds) if and only if $P_{comp} \neq 0$. By definition, $P_{comp} \neq 0$ if and only if Ω' has positive measure (i.e. (ii) holds). As Ω' has either measure 0 or meassure 1, it has positive measure if and only if it has measure 1 (i.e. (iii) holds). □

6. Some deterministic approximation results

In this section we present three deterministic results. The first two results give estimates for the difference

$$\chi_\Lambda \phi(H_\omega) - \phi(p_{\Lambda(\omega)} H_\omega \, i_{\Lambda(\omega)})$$

for suitable functions ϕ on $\mathbb{R}$ and $\Lambda \subset X$. In one way or other this type of result is entering all considerations on convergence of the IDS (see the first list of references in the Introduction). On the technical level our arguments are related to [10, 13, 35].

The last result gives a lemma from measure theory on convergence of distribution functions for measures on $\mathbb{R}$. This type of lemma seems to have first been used in the present context in [12]. There, one can also find a proof. For the convenience of the reader we include an alternative proof (and actually give a slightly strengthened result).

Lemma 6.1. *Let* (I_n) *be a Følner sequence in* Γ *and* $\Lambda_n := I_n F'$ *as before. Let* H *satisfy (A). Let* ϕ *be a continuous function on* $\mathbb{R}$. *Then, for any* $\omega \in \mathcal{D}$

$$\frac{|\operatorname{Tr}(\chi_{\Lambda_n} \phi(H_\omega)) - \nu_{\omega,n}(\phi)|}{|I_n|} \longrightarrow 0$$

for $n \to \infty$.

Proof. First we prove that the statement holds if $\phi(x) = x^k$ for some $k \in \mathbb{N}$. Note that $\nu_{\omega,n}(\phi) = \operatorname{Tr}(H_{\omega,n}^k)$ is a sum over closed paths of lenght k, more precisely

$$\sum_{x \in \Lambda_n(\omega)} \sum_{x_1,\ldots,x_{k+1} \in \Lambda_n(\omega)} H_{\omega,n}(x_1, x_2) \ldots H_{\omega,n}(x_k, x_{k+1})$$

where in the second sum $x_1 = x = x_{k+1}$. Since $\operatorname{Tr}(\chi_{\Lambda_n} H_\omega^k)$ can be written in a similar way, the difference $\operatorname{Tr}(\chi_{\Lambda_n} H_\omega^k) - \operatorname{Tr}(H_{\omega,n}^k)$ is bounded in modulus by

$$\sum_{x \in \partial^{kR} \Lambda_n(\omega)} \sum_{x_1,\ldots,x_{k+1} \in X(\omega)} |H_{\omega,n}(x_1, x_2)| \ldots |H_{\omega,n}(x_k, x_{k+1})|$$

$$\leq \sum_{x \in \partial^{kR} \Lambda_n(\omega)} \omega(B_{kR})^k \|H\|^k \leq \omega(\partial^{kR} \Lambda_n)(M_{kR})^k \|H\|^k.$$

Here R denotes the finite hopping range of the operator H_ω. Now one uses Proposition 3.4 and the fact that $I_n, n \in \mathbb{N}$ is a Følner sequence to conclude that

$$\lim_{n \to \infty} \frac{\omega(\partial^{kR} \Lambda_n)}{|I_n|} (M_{kR})^k \|H\|^k = 0$$

Now the following considerations extend the convergence result to all $\phi \in C(\mathbb{R})$:

Note that the only relevant data of the function ϕ are its values on the spectrum of H_ω, which is a compact set as H_ω is bounded. Let $\mathcal{S}$ be the family of functions for which the statement of the lemma holds. We just proved that all polynomials belong to $\mathcal{S}$. Moreover, the set $\mathcal{S}$ is obviously closed under uniform convergence. The statement follows from Stone-Weierstrass theorem. □

Lemma 6.2. *Let* (I_n) *be a Følner sequence in* Γ *and* $\lambda \in \mathbb{R}$ *arbitrary. Then, for any* $\omega \in \tilde{\mathcal{D}}$

$$\frac{|\operatorname{Tr}(\chi_{\Lambda_n} E_\omega(\{\lambda\})) - \nu_{\omega,n}(\{\lambda\})|}{|I_n|} \longrightarrow 0$$

for $n \to \infty$.

Proof. Let R be the hopping range of H. Fix $\omega \in \mathcal{D}$. Let V_n be the subspace of all solutions of $(H_\omega - \lambda)v = 0$, which vanish outside $\Lambda_{n,R}$. Let D_n be the dimension of V_n.

We now apply Lemma 3.5 to the space U of all solutions of $(p_{\Lambda_n(\omega)}H_\omega i_{\Lambda_n(\omega)} - \lambda)u = 0$ and note that $U_R = V_n$ as the hopping range of H_ω is R. This gives

$$0 \leq \nu_{\omega,n}(\{\lambda\}) - D_n \leq \omega(\partial^R(\Lambda_n)).$$

Moreover, obviously,

$$D_n \leq \mathrm{Tr}(\chi_{\Lambda_n}E_\omega(\{\lambda\})) \leq \dim(\mathrm{ran}(\chi_{\Lambda_n}E_\omega(\{\lambda\}))).$$

We now apply Lemma 3.5 to $U' := \mathrm{ran}\,\chi_{\Lambda_n}E_\omega(\{\lambda\})$ and note that $U'_R = V_n$ as the hopping range of H_ω is R. This gives

$$0 \leq \mathrm{Tr}(\chi_{\Lambda_n}E_\omega(\{\lambda\})) - D_n \leq \omega(\partial^R(\Lambda_n)).$$

By Proposition 3.2, there exists $\rho > 0$ with $\partial^R(\Lambda_n) \subset (\partial^\rho I_n)F'$ for all $n \in \mathbb{N}$. The statement follows now from the triangle inequality and Proposition 3.4. $\qquad\square$

Lemma 6.3. *Let ν be a probability measure on $\mathbb{R}$. Let (ν_n) be a sequence of bounded measures on $\mathbb{R}$ which satisfy*

- *ν_n converge weakly to the measure ν,*
- *$\nu_n(\{\lambda\}) \longrightarrow \nu(\{\lambda\})$ for all $\lambda \in \mathbb{R}$.*

Then, the distribution functions $\lambda \mapsto \nu_n((-\infty, \lambda])$ of the ν_n converge with respect to the supremum norm to the distribution function $\lambda \mapsto \nu((-\infty, \lambda])$ of ν.

Remark. Note that vague convergence $\nu_n \to \nu$, actually implies weak convergence, if one of the two following conditions hold:

- all ν_n are probability measures, or
- there exists a compact interval such that the supports of ν and of ν_n is contained in this interval for all $n \in \mathbb{N}$.

Proof. Let $\epsilon > 0$ be arbitrary. Let ν_c denote the continuous part of ν and ν_p the point part of ν. Choose

$$-\infty = t_{-1} = t_0 < t_1 < \ldots < t_L < t_{L+1} < t_{L+2} = \infty$$

such that

(∗) $\nu_c((t_j, t_{j+1}]) \leq \epsilon$ for $j = 1, \ldots L$, $\nu(-\infty, t_1) \leq \epsilon$ and $\nu(t_{L+1}, \infty) \leq \epsilon$.

Such a choice is possible since ν is a probability measure. Let (λ_k) be an enumeration of the points of discontinuity of ν. Assume without loss of generality that k runs through all of $\mathbb{N}$. Choose $N \in \mathbb{N}$ with

(∗∗) $\sum_{k=N+1}^\infty \nu(\{\lambda_k\}) \leq \epsilon$.

Choose continuous functions ϕ_j with $\chi_{(-\infty,t_j]} \leq \phi_j \leq \chi_{(-\infty,t_{j+1}]}$ for $j = 0, \ldots, L$. Set $\phi_{-1} \equiv 0$. For all large enough n we then have

(∗∗∗) $|\nu_n(\phi_j) - \nu(\phi_j)| \leq \epsilon$ for $j = 1, \ldots, L+1$,
(∗∗∗∗) $|\nu_n(\chi_{\{\lambda_j | j=1,\ldots,N\}}) - \nu(\chi_{\{\lambda_j | j=1,\ldots,N\}})| \leq \epsilon$.

For such n we prove now

$$\nu((-\infty, \lambda]) - \nu_n((-\infty, \lambda]) \leq 5\epsilon$$

for all $\lambda \in \mathbb{R}$ as follows: Choose $j \in 0, \ldots, L+1$ with $t_j \leq \lambda < t_{j+1}$ and define ψ by

$$\chi_{(-\infty,\lambda]} = \phi_{j-1} + \psi$$

be given. Since $0 \leq \psi \leq 1$ on $\mathbb{R}$ and $\mathrm{supp}\,\psi \subset [t_{j-1}, t_{j+1}]$ we then obtain

$$
\begin{aligned}
\nu_n((-\infty, \lambda]) &= \nu_n(\phi_{j-1}) + \nu_n(\psi) \\
(***) \quad &\geq \nu(\phi_{j-1}) - \epsilon + \nu_n(\psi) \\
&= \nu((-\infty, \lambda]) - \nu(\psi) - \epsilon + \nu_n(\psi) \\
(*) \quad &\geq \nu((-\infty, \lambda]) - \nu_p(\psi) - 3\epsilon + \nu_n(\psi) \\
&= \nu((-\infty, \lambda]) - 3\epsilon + \nu_n(\psi) - \nu_p(\psi) \\
(**) \quad &\geq \nu((-\infty, \lambda]) - 4\epsilon + \nu_n(\psi) - \nu_p(\psi \chi_{\{\lambda_j | j=1,\dots,N\}}) \\
&\geq \nu((-\infty, \lambda]) - 4\epsilon + \nu_n(\psi \chi_{\{\lambda_j | j=1,\dots,N\}}) - \nu_p(\psi \chi_{\{\lambda_j | j=1,\dots,N\}}) \\
(****) \quad &\geq \nu((-\infty, \lambda]) - 5\epsilon.
\end{aligned}
$$

Note that the above inequalities hold also if we replace $\nu((-\infty, \lambda]), \nu_n((-\infty, \lambda])$ by $\nu((-\infty, \lambda)), \nu_n((-\infty, \lambda))$. Thus we have proven

$$
\lim_{n \to \infty} \sup_{\lambda \in \mathbb{R}} \nu((-\infty, \lambda]) - \nu_n((-\infty, \lambda]) \leq 0
$$

To prove the opposite inequality we use the measure $\tilde{\nu}$ which is the reflection of ν around the origin. It inherits all properties of ν which have been used in the previous calculation. Then

$$
\tilde{\nu}((-\infty, \lambda)) = \nu([-\lambda, \infty)) = 1 - \nu((-\infty, -\lambda])
$$

The above argument yields $\lim_{n\to\infty} \sup_{\lambda \in \mathbb{R}} \tilde{\nu}((-\infty, \lambda)) - \tilde{\nu}_n((-\infty, \lambda)) \leq 0$. Since $\tilde{\nu}((-\infty, \lambda)) - \tilde{\nu}_n((-\infty, \lambda)) = \nu_n((-\infty, -\lambda]) - \nu((-\infty, -\lambda])$, the proof is completed. $\qquad\square$

7. PROOF OF THEOREM 2.4

In this section, we prove Theorem 2.4. Since here ergodicity plays a role, we first discuss various consequences of ergodic theorems.

For a function v on Ω and $I \subset \Gamma$ with $|I| > 0$ we define the function v_I on Ω by

$$
v_I(\omega) := \int_I v(\gamma^{-1}\omega) d\gamma.
$$

For any v integrable with respect to μ and any tempered Følner sequence (I_n) there exists a Γ-invariant $\bar{v} \in L^1(\mu)$ such that

$$
\lim_{n \to \infty} \frac{1}{|I_n|} v_{I_n}(\omega) = \bar{v}(\omega)
$$

for almost every $\omega \in \Omega$ and in L^1, see [33]. Moreover, if μ is ergodic $\bar{v}(\omega) = \int v(\omega) d\mu(\omega)$ almost surely.

Our natural setting is not concerned with v_I but rather with functions of the form $\chi_{IF'}$. We next show that these two types of functions are comparable. To do so, we recall that we have fixed a point $p \in F' \subset X$.

Proposition 7.1. *Let u be a measurable nonnegative function on X satisfying* (1). *Let $r > 0$ such that the support of u is contained in the open ball $B_r(p)$ around p with radius r. Then,*

$$
|\chi_{IF'}(x) - u_I(x)| \leq \chi_{\partial^r(IF')}(x)
$$

for any $x \in X$.

Proof. Consider first $x \in (IF')_r$: Then, $B_r(x) \subset IF'$. Any $\gamma \in \Gamma$ with $u(\gamma^{-1}x) \neq 0$ satisfies $d(\gamma^{-1}x, p) < r$, hence

$$
\gamma p \in B_r(x) \subset IF'
$$

and thus $\gamma \in I$. For such x we therefore obtain

$$
\int_I u(\gamma^{-1}x) d\gamma = \int u(\gamma^{-1}x) d\gamma = 1.
$$

Consider now $x \notin (IF')^r$: Then, $B_r(x) \subset X \setminus IF'$. Any $\gamma \in \Gamma$ with $u(\gamma^{-1}x) \neq 0$ satisfies

$$
\gamma p \in B_r(x) \subset X \setminus (IF')
$$

and hence $\gamma \notin I$. For such x we therefore obtain

$$\int_I u(\gamma^{-1}x)d\gamma = 0.$$

Consider now $x \in (IF')^r \setminus (IF')_r$: As u is nonnegative with $\int u(\gamma^{-1}x)d\gamma = 1$, we obtain $0 \leq u(x) \leq 1$ for such u.

Having considered these three cases, we can easily obtain the statement. $\square$

We now derive two consequences of the previous proposition and the ergodic theorem.

Proposition 7.2. *Let (I_n) a tempered Følner sequence in Γ and $T \in \mathcal{N}$ be arbitrary. Then, for almost every $\omega \in \Omega$,*

$$\lim_{n\to\infty} \frac{1}{|I_n|} \mathrm{Tr}(\chi_{\Lambda_n}T_\omega) = g(\omega).$$

where $g \in L^1(\mu)$ is Γ-invariant. The convergence holds also in L^1 sense. If μ is ergodic, $g = \mathrm{dens}(m)\tau(T)$ almost surely.

Proof. We show that $\omega \mapsto \frac{1}{|I_n|} \mathrm{Tr}(\chi_{\Lambda_n}T_\omega)$ converges almost surely and in L^1 for $n \to \infty$. By Proposition 3.4 combined with Proposition 7.1, it suffices to consider the sequence

$$\mathrm{Tr}(u_{I_n}T_\omega) = \sum_{x \in X(\omega)} u_{I_n}(x)T_\omega(x,x)$$

with u satisfying (1), instead of considering

$$\mathrm{Tr}(\chi_{\Lambda_n}T_\omega) = \sum_{x \in \Lambda_n(\omega)} T_\omega(x,x)$$

Define with such a u

$$v(\omega) := \mathrm{Tr}(u\,T_\omega) = \sum_{x \in X(\omega)} u(x)T_\omega(x,x).$$

By the ergodic theorem, $v_{I_n}/|I_n|$ converges almost surely to some Γ-invariant $g \in L^1(\mu)$. A direct calculation using equivariance shows

$$v_{I_n}(\omega) = \sum_{x \in X(\omega)} u_{I_n}(x)T_\omega(x,x).$$

Thus the first statement of the Proposition is proven. If μ is ergodic, then $g = \int \mathrm{Tr}(u\,T_\omega)d\mu(\omega) = \mathrm{dens}(m)\tau(T)$ almost surely. $\square$

We note the following special case of the previous proposition.

Corollary 7.3. *Let μ be ergodic and (I_n) be a tempered Følner sequence in Γ. For almost every $\omega \in \Omega$, we have $\lim_{n\to\infty} \frac{\omega(\Lambda_n)}{|I_n|} = \mathrm{dens}(m) = \mathrm{dens}(m)\tau(\mathrm{Id})$. In particular, $\tau(\mathrm{Id}) = 1$.*

Proof. The previous proposition with $T = \mathrm{Id}$ shows pointwise and L^1 convergence of the functions $\omega \mapsto \frac{\omega(\Lambda_n)}{|I_n|}$ to the constant $\mathrm{dens}(m)\tau(\mathrm{Id})$. Since

$$\mathrm{dens}(m) = \frac{m(\Lambda_n)}{|I_n|} = \int \frac{\omega(\Lambda_n)}{|I_n|}d\mu(\omega)$$

for all $n \in \mathbb{N}$, $\tau(\mathrm{Id})$ must be equal to 1 and the statement follows. $\square$

Proof of Theorem 2.4. Since $\frac{N_{\omega,n}}{\omega(\Lambda_n)} = \frac{N_{\omega,n}}{|I_n|}\frac{|I_n|}{\omega(\Lambda_n)}$ and $\frac{|I_n|}{\omega(\Lambda_n)}$ converges to $\mathrm{dens}(m)^{-1}$ almost surely, it suffices to show convergence of $\frac{N_{\omega,n}}{|I_n|}$. There are at most countably many points of discontinuity of ν_H. Thus, Lemma 6.2 combined with Proposition 7.2, gives convergence in all points of discontinuity of ν_H almost surely. On the other hand, note that the space of continuous functions with compact support on $\mathbb{R}$ is separable. Thus, weak convergence of probability measures follows from convergence on a countable dense subset of continuous functions on $\mathbb{R}$ with compact support. Thus, Lemma 6.1 combined with Proposition 7.2 gives weak convergence of the measures almost surely. Now, the result follows from Lemma 6.3. $\square$

Remark 7.4. Even in the case that μ is not ergodic we can prove a uniform convergence statement. By assumption (A) there is an $R' \in \mathbb{R}$ such that $\sigma(H_\omega) \subset [-R', R']$. Let $\psi_j, j \in \mathbb{N}$ be a countable dense set in $C([-R', R'])$. By Proposition 7.2 there exists a set of full measure $\Omega_j \subset \Omega$ such that for all $\omega \in \Omega_j$

$$l_\omega(\psi_j) := \lim_{n \to \infty} \frac{1}{|I_n|} \operatorname{Tr}(\chi_{\Lambda_n} \psi_j(H_\omega))$$

exists. This way one defines for all $\omega \in \tilde{\Omega} := \cap_{j \in \mathbb{N}} \Omega_j$ a positive, bounded linear functional l_ω, i.e. a measure. In particular one may define a density for such ω by $\operatorname{dens}(\omega) = \lim_{n \to \infty} \frac{\omega(\Lambda_n)}{|I_n|}$. For $\omega \in \tilde{\Omega}$ with $\operatorname{dens}(\omega) > 0$ we can as before renormalize the sequence $\frac{N_{\omega,n}}{\omega(\Lambda_n)} = \frac{N_{\omega,n}}{|I_n|} \frac{|I_n|}{\omega(\Lambda_n)}$ which thus converges to the distribution function of $\nu_\omega := \frac{l_\omega}{\operatorname{dens}\omega}$. If $\operatorname{dens}(\omega) = 0$ we still see that $\operatorname{Tr}(\chi_{\Lambda_n} \psi(H_\omega)) \leq \|\psi\|_\infty \omega(\Lambda_n)$. Thus if $X(\omega)$ is not empty

$$\frac{\operatorname{Tr}(\chi_{\Lambda_n} \psi(H_\omega))}{\omega(\Lambda_n)} \leq \|\psi\|_\infty$$

and we can conlude by the Banach-Alaouglu theorem that there is a subsequence along which $\frac{N_\omega^{n_k}}{\omega(\Lambda_{n_k})}$ converges weakly. A posteriori, we can enhance this to uniform convergence using Lemma 6.3.

8. Models with aperiodic order

In this section, we discuss models with aperiodic order. In these cases $\Gamma = X = \mathbb{R}^m$ is a continuous group. We recover the main result of [21] concerning characterization of jumps of the IDS via compactly supported eigenfunctions. In fact, we obtain a strengthening of the result of [21] in three respects: We do not need an ergodicity assumption anymore, we do not need a finite local complexity assumption and we identify the size of the jumps as a equivariant dimension. (Note that the latter, however, can directly derived from the convergence statement in [32]). We also obtain a result on convergence of the IDS. This result, however, is strictly weaker than the results of [32] as it neither holds for all $\omega \in \Omega$ nor gives an explicit error bound on the speed of convergence.

The setting is as follows: There is an obvious action of $\Gamma = \mathbb{R}^m$ on $X = \mathbb{R}^m$ by translation. A fundamental domain is given by the compact set $\{0\}$ and the map $\Phi \colon \Gamma \longrightarrow \mathbb{R}^m$ is just the identity and therefore an isometry. Hence, the geometric assumptions of our setting are satisfied. The ball around $x \in \mathbb{R}^m$ with radius r is denoted by $B_r(x)$. As before, $\mathcal{D}$ is the set of subsets of $\mathbb{R}^m$ whose elements have Euclidian distance at least 1. We call a subset $\mathcal{M}$ of $\mathcal{D}$ a collection of Delone sets if the following holds:

– There exists an $R' > 0$ with $A \cap B_{R'}(x) \neq \emptyset$ for every $A \in \mathcal{M}$ and $x \in X$.

It is said to be of finite local complexity if it also satisfies the following:

– For each $r > 0$, the set $\{(A - x) \cap B_r(0) \mid A \in \mathcal{M}, x \in A\}$ is finite.

Let $\mu = \mu_\mathcal{M}$ be an invariant probability measure on $\mathcal{D}$ whose support Ω is a collection of Delone sets. In particular $X(\omega) = \omega \in \Omega$ is a discrete subset of $\mathbb{R}^d$. Assume that the density of m is 1.

This setting gives a notion of an equivariant operator as a family (H_ω) of operators $H_\omega \colon \ell^2(X(\omega)) \longrightarrow \ell^2(X(\omega))$ with

$$H_{\gamma+\omega}(\gamma + x, \gamma + y) = H_\omega(x, y).$$

The natural trace is defined via

$$\tau(H) = \int_\Omega \operatorname{Tr}(uH_\omega) d\mu(\omega),$$

where the continuous $u \colon \mathbb{R}^m \longrightarrow \mathbb{R}$ is an arbitrary function with compact support and $\int_{\mathbb{R}}^m u(x)dx = 1$.

For an operator satisfying (A) and $\lambda \in \mathbb{R}$ our abstract results give:

(i) Let U_ω be the subspace of $\ell^2(X(\omega))$ spanned by compactly supported solutions to $(H_\omega - \lambda)u = 0$. Then, the value $\nu_H(\{\lambda\}) = N_H(\lambda) - N_H(\lambda-) = \tau(E_H(\{\lambda\}))$ is the equivariant dimension of the subspace $\int^\oplus U_\omega d\mu(\omega)$ of $\int_\Omega^\oplus \ell^2(X(\omega))d\mu(\omega)$.

(ii) If μ is furthermore assumed ergodic, then λ is a point of discontinuity of N_H if and only if there exists a compactly supported eigenfunction of H_ω to λ for almost every $\omega \in \Omega$. In this case, these compactly supported eigenfunctions span the eigenspace of H_ω to λ.

(iii) The normalized finite volume counting functions $\frac{N_{\omega,n}}{|\Lambda_n|}$ converge almost surely with respect to the supremum norm towards the function N_H, i.e.

$$\lim_{n\to\infty} \left\| \frac{1}{|\Lambda_n|} N_{\omega,n} - N_H \right\|_\infty = 0.$$

9. Periodic models on amenable graphs and CW-complexes

In this section, we briefly discuss how the corresponding results of [6, 8, 10, 23, 35, 36] can be recovered in our framework. In these cases the group Γ as well as the space X are discrete and countable.

The geometric setting in the cited works is given either by a graph or a CW-complex. Let us first consider the case that a graph $G = (V, E)$ with vertex set V and edge set E and a finitely generated amenable group Γ are given, such that each vertex degree is finite and Γ acts freely and cocompactly on G by automorphisms. We set $X := V$ and show that the assumptions of our setting are satisfied: A finite set S_0 of generators of Γ which is symmetric in the sense that $S_0 = S_0^{-1} := \{\gamma^{-1} \mid \gamma \in S_0\}$ defines a word metric d_{S_0} on Γ. Obviously, d_{S_0} is invariant under the action of Γ on itself. If we choose a different (finite, symmetric) set of generators S it gives rise to another metric d_S. This metric is equivalent to the metric d_{S_0} and in particular the two spaces (Γ, d_S) and (Γ, d_{S_0}) are roughly isometric.

As Γ acts cocompactly, there exists a compact (i.e. finite) fundamental domain F' of the action of Γ. In particular, F' is equal to its closure F. As Γ acts freely, we obtain a well defined map

$$\Phi \colon X \longrightarrow \Gamma, \text{with } x \in \Phi(x)F.$$

We have to show hat this map is a rough isometry. To do so, we need of course a metric on X. Let us first assume that the graph is connected. Then $X = V$ carries a natural metric d coming from finite paths between the points. This metric is obviously Γ-invariant. Set

$$C := \max_{s \in S \cup \{\text{id}\}} \{d(x,y) \mid x \in F, y \in sF\} < \infty.$$

Then,

$$d(x,y) \le C d_\Gamma(\Phi(x), \Phi(y)) + C.$$

As for the converse inequality, we need some more preparation. We say that $\gamma \in \Gamma$ is a neighbor of $\rho \in \Gamma$ if there exists an edge connecting a vertex in ρF with a vertex in γF. Denote the set of neighbors of $\text{id} \in \Gamma$ by S_0. Since X is connected, S_0 is a set of generators of Γ; since each vertex degree is finite, S_0 is finite; and by the properties of the action of Γ, S_0 is symmetric. Thus

$$d_{S_0}(\Phi(x), \Phi(y)) \le d(x,y).$$

Since any word metric on Γ is roughly isometric with d_{S_0}, this shows that Φ is indeed a rough isometry.

If the graph is not connected, there is no natural choice of a metric on X. In this case, we can induce a metric on X by the metric on Γ and Φ in two steps: the metric on the group Γ defines a distance between different fundamental domains. We can assume that this distance function takes values in $\mathbb{N}$. Within the fundamental domains the distance between two points is defined using shortest paths. Let us scale the latter distance function such that the diameter of a fundamental domain is bounded by one. This way one obtains a metric on X which is by construction roughly isometric to Γ.

Let us now turn to the case of CW-complexes. Thus let a CW-complex Y and a finitely generated amenable group Γ be given which acts freely on Y by automorphisms. We assume that the quotient Y/Γ is a CW-complex of finite type, i.e. all its skeleta are finite. For a $j \in \mathbb{N}$ denote

by Y_j the set of j-cells in Y. Two such cells are called adjacent if either the intersection of their closures contains a $j - 1$ cell of Y, or if both are contained in a the closure of a single $j + 1$ cell. Since we assumed that the quotient Y/Γ is of finite type, the number of cells adjacent to any given cell is finite. Now we fix $j \in \mathbb{N}$ and define a graph G_j with vertex set $V = Y_j$. Two elements V are connected by an edge iff they are adjacent. Each automorphism of the original CW-complex induces a graph-automorphism on G_j. In particular, Γ acts freely and cocompactly on G_j. Thus we are back in the setting which we discussed at the start of this section. (Note that for each $j \in \mathbb{N}$ we extract from the CW-complex Y a different graph G_j and correspondingly the graph Hamiltonians, which we define below, will also depend on j.)

These considerations show that the geometric assumptions of our model are satisfied in the cited works.

In the present setting the set X itself belongs to $\mathcal{D}$ and is in fact invariant under the action of Γ. Thus, we can choose the measure μ on $\mathcal{D}$ to be supported on $\{X\}$. This means that Ω consists of a single element which is just X. Thus, everything depending on the family $\omega \in \Omega$ is replaced by a single object in the sequel. In particular, we certainly have all ergodic assumptions satisfied. In fact, all statements concerning almost sure convergence in Ω give deterministic statements.

We will now deal with the operator theoretical side of things. In [36, 35] one is given a family $s_\gamma, \gamma \in \Gamma$ of maps $s_\gamma \colon V \longrightarrow \{z \in \mathbb{C} \mid |z| = 1\}$. In the other cases one just sets $s_\gamma \equiv 1, \gamma \in \Gamma$. The family of operators in question is then given by a single operator $H = H_X$ satisfying

$$s_\gamma(x) H(\gamma x, \gamma y) \overline{s_\gamma(y)} = H(x, y).$$

The natural trace becomes

$$\tau(H) = \mathrm{Tr}(\chi_F H).$$

As before denote by H_n the restriction of the operator H to $\Lambda_n = I_n F$, where $I_n \in \Gamma$ is a Følner sequence. With the usual convention $N_H(\lambda) := \tau(E_H(\lambda))$ and $N_{H,n} := \mathrm{Tr}\, E_{H_n}(\lambda)$, our results can the be re formulated as follows:

(i) For any $\lambda \in \mathbb{R}$, the value $\nu_H(\{\lambda\}) = \tau(E_H(\{\lambda\}))$ is the equivariant dimension of the subspace of $\ell^2(X)$ spanned by compactly supported solutions of $(H_\omega - \lambda)u = 0$.

(ii) A number $\lambda \in \mathbb{R}$ is a point of discontinuity of N_H if and only if there exists a compactly supported eigenfunction of H to λ.

(iii) The functions $N_{H,n}$ converge with respect to the supremum norm towards the function N_H, i.e.

$$\lim_{n \to \infty} \left\| \frac{1}{|\Lambda_n|} N_{H,n} - N_H \right\|_\infty = 0.$$

10. Anderson and percolation Hamiltonians

In this section we discuss the application of our results to certain types of random models on graphs. More precisely, we consider Anderson models, random hopping models, as well as site and bond percolation models. They can be understood as randomized versions of the operators introduced in Section 9. In particular we are again given a graph $G = (V, E)$ with bounded vertex degree on which a finitely generated, amenable group Γ acts freely and cocompactly by automorphisms.

The application of our abstract theorems to these models recover and extend in particular the results which concern the construction of the IDS by its finite volume analogues obtained in [44, 20, 27].

Let us introduce the *Anderson-Percolation Hamiltonian.* We set $X = V$ and assume that a function $h_0 \colon X \times X \to \mathbb{C}$ is given with $h_0(x, y) = \overline{h_0(y, x)}$ and $h_0(x, y) \neq 0 \Rightarrow d(x, y) < R$. Consider the setting from Section 2 and assume additionally that there exist a constant C and a function $\mathcal{V}_\omega \colon X \times X \to [-C, C]$ with support on the diagonal $D := \{(x, x) \mid x \in X\}$ sucht that for all $\omega = (A, h) \in \Omega$

$$h = (h_0 + \mathcal{V}_\omega) \chi_{A \times A}.$$

The operator associated to $\omega \in \Omega$ acts on the ℓ^2 space of the diluted graph $X(\omega) = A$. More precisely for each $v \in \ell^2(X(\omega))$ and $x \in X(\omega)$

$$(H_\omega v)(x) = \Big(\sum_{y \in X(\omega)} h_0(x, y)v(y) \Big) + \mathcal{V}_\omega v(x).$$

We can think of the first term as the kinetic energy or hopping term and of the second as a random potential. In the special case that $X(\omega) = X$ for all $\omega \in \Omega$ and

$$h_0(x, y) = \chi_1(d(x, y)) := \begin{cases} 1 & \text{if } d(x, y) = 1, \\ 0 & \text{otherwise} \end{cases}$$

we obtain the Anderson model. Likewise, for $\mathcal{V}_\omega \equiv 0$ and $h_0(x, y) = \chi_1(d(x, y))$ we have the site-percolation Hamiltonian. Of course it is possible to choose the measure μ in such a way that the discussed models are i.i.d. with respect to the coordinates $x \in X$, see cf. [45].

Let us now discuss the random hopping model, which includes the bond percolation model as a special case. Such models have bee considered for instance in [22, 20]. Here we require each $(A, h) \in \Omega$ to satisfy additionally to the conditions in Section 2 that $A = X$ and $h \equiv 0$ on the diagonal D. The matrix coefficient $h(x, y)$ may be considered as a hopping term between x and y (at least when h is non-negative). In the case that $h(x, y) \in \{0, 1\}$ we obtain a bond-percolation Hamiltonian. Again, a suitable choice of the measure μ yields an i.i.d. model.

Note that Dirichlet and Neumann boundary terms as considered in [20, 1] can be incorporated into a potential energy term $\mathcal{V}_\omega$. Let us emphasize that the models discussed above do not have necessarily finite local complexity.

As before denote by $N_{\omega,n}$ the eigenvalue counting functions associated to a Følner sequence I_n in Γ. Our results from Section 2 can the be now reformulated as follows:

(i) Let U_ω be the subspace of $\ell^2(X(\omega))$ spanned by compactly supported eigenfunctions of H_ω Then, $\nu_H(\{\lambda\}) = N_H(\lambda) - \lim_{\epsilon \to 0} N_H(\lambda - \epsilon)$ equals the equivariant dimension of the subspace $\int^\oplus U_\omega d\mu(\omega)$ of $\int_\Omega^\oplus \ell^2(X(\omega))d\mu(\omega)$.

(ii) If μ is furthermore assumed ergodic, then λ is a point of discontinuity of N_H if and only if there exist compactly supported eigenfunctions to H_ω and λ for almost every $\omega \in \Omega$. In this case, these eigenfunctions actually span the eigenspace of H_ω to the eigenvalue λ for almost every $\omega \in \Omega$.

(iii) For μ-almost all ω, the distibution function N_H can be approximated by $N_{\omega,n}$ uniformly in the energy variable :

$$\lim_{n \to \infty} \Big\| \frac{1}{|\Lambda_n|} N_{\omega,n} - N_H \Big\|_\infty = 0.$$

11. PERCOLATION ON DELONE SETS

Consider the setting explained in Section 8. In particular, let the space X and the group Γ equal $\mathbb{R}^d$. Let $\mathcal{M} \subset \mathcal{D}$ be a collection of Delone sets of finite local complexity and $h_0 \colon \mathbb{R}^d \to \mathbb{R}$ a bounded, measurable function of compact support satisfying $h_0(-x) = h_0(x)$. For each $A \in \mathcal{M}$ let $\mathcal{E}(A)$ consist of pairs of subsets E_1, E_2 of $A \times A$ satisfying the following

(1) E_1 and E_2 are disjoint,
(2) $E_i \cap D = \emptyset$ where as before $D = \{(x, x) \mid x \in \mathbb{R}^d\}$ and $(x, y) \in E_i \Rightarrow (y, x) \in E_i$ for $i \in \{1, 2\}$.

In other words E_1, E_2 is a pair of disjoint sets of edges for the vertex set A. For such $(E_1, E_2) \in \mathcal{E}(A)$ set

$$Z_{A, E_1, E_2} := \{(A, h) \mid h \colon A \times A \to \mathbb{R}, h(x, y) \in \{0, h_0(x - y)\},$$
$$h(x, y) = h_0(x - y) \text{ for all } (x, y) \in E_1, h(x, y) = 0 \text{ for all } (x, y) \in E_2\}.$$

For $p \in [0, 1]$ fixed, define a measure μ_A on the cylinder sets Z_{A, E_1, E_2} with E_1 and E_2 finite by setting

$$\mu_A(Z_{A, E_1, E_2}) := p^{|E_1|}(1 - p)^{|E_2|}$$

and extend it to $\{h\colon A \times A \to \mathbb{R}\}$ by uniqueness. Recall that the projection

$$\pi_1 \colon \tilde{\mathcal{D}} \to \mathcal{D}, \quad (A, h) \mapsto A$$

is measurable. Denote by $\mu_{\mathcal{M}}$ an invariant probability measure on $\mathcal{D}$ whose support is $\mathcal{M}$. Next we define a measure μ on $\tilde{\mathcal{D}}$. For this purpose we denote the pairs (A, h) by ω althought the measure μ and thus its support Ω are yet to be identified. For a mesurable $B \subset \tilde{\mathcal{D}}$ we define

$$\mu(B) = \int_{\mathcal{M}} \left(\int_{\pi^{-1}(A)} \chi_B \, d\mu_A(\omega) \right) d\mu_{\mathcal{M}}(A).$$

Note that if E_1 and E_2 form a partition of $A \times A$, then Z_{A,E_1,E_2} contains a single element. Thus every $\omega = (A, h)$ in the support Ω of μ can be identified with such an Z_{A,E_1,E_2}. The associated operator H_ω has matrix coefficients

$$H_\omega(x, y) = \begin{cases} h_0(x - y) = h_0(y - x) & \text{if } (x, y) \in E_1 \\ 0 & \text{if } (x, y) \in E_2 \end{cases}$$

and defines a bond-percolation Hamiltonian on $\mathcal{M}$.

Since we are in Euclidean space we can choose $I_n = \Lambda_n$ to be balls or cubes of diameter $n \in \mathbb{N}$.

Again we have the following results:

(i) Let U_ω be the subspace of $\ell^2(X(\omega))$ spanned by compactly supported solutions of $(H_\omega - \lambda)u = 0$. Then, $\nu_H(\{\lambda\}) = N_H(\lambda) - \lim_{\epsilon \to 0} N_H(\lambda - \epsilon)$ equals the equivariant dimension of the subspace $\int^\oplus U_\omega d\mu(\omega)$ of $\int_\Omega^\oplus \ell^2(X(\omega)) d\mu(\omega)$.

(ii) If μ is furthermore assumed ergodic, then λ is a point of discontinuity of N_H if and only if there exists a compactly supported eigenfunctions to H_ω and λ for almost every $\omega \in \Omega$. In this case, these eigenfunctions actually span the eigenspace of H_ω to the eigenvalue λ for almost every $\omega \in \Omega$.

(iii) The distribution function N_H can be approximated by $N_{\omega,n}$ uniformly in the energy variable:

$$\lim_{n \to \infty} \left\| \frac{1}{|\Lambda_n|} N_{\omega,n} - N_H \right\|_\infty = 0$$

almost surely.

Appendix A. Topology on $\tilde{\mathcal{D}}$ and compactness

We discuss the topology and compactness of $\tilde{\mathcal{D}}$.

We start with a slightly more general setting. Let Z be a locally compact space. Denote the set of measures on Z by $M(Z)$ and the set of continuous functions with compact support on Z by $C_c(Z)$. The set $M(Z)$ can be embedded into $\prod_{\varphi \in C_c(Z)} \mathbb{C}$ via

$$\Psi \colon M(Z) \longrightarrow \prod_{\varphi \in C_c(Z)} \mathbb{C}, \quad \Psi(\mu) = (\varphi \mapsto \mu(\varphi)).$$

The product topology then induces the initial topology on $M(Z)$, which is called vague topology. Assume now that $\mathcal{U}$ an open covering of Z. Define for $C > 0$ the set

$$M_{C,\mathcal{U}} := M_{C,\mathcal{U}}(Z) = \{\mu \in M(Z) : \mu(U) \le C \text{ for all } U \in \mathcal{U}\}.$$

As $\mathcal{U}$ is an open covering, the support of any $\varphi \in C_c(Z)$ can be covered by finitely many elements of $\mathcal{U}$. Then, Tychonoff theorem easily yields that $M_{C,\mathcal{U}}$ is contained in a compact subset of $M(Z)$. As $M_{C,\mathcal{U}}$ is closed it must then be compact as well.

We now specialize these considerations to the situation outlined in Section 2:

Thus, we are given a locally compact metric space X. As before, the set of uniformly discrete subsets of X with minimal distance 1 is denoted by $\mathcal{D}$. Let Y be a compact space and set

$Z = X \times Y$. Let $\mathcal{D}_Y$ be the set of all functions $f \colon A \longrightarrow Y$ with $A \in \mathcal{D}$. We can identify elements of $\mathcal{D}_Y$ with measures on Z via

$$\delta \colon \mathcal{D}_Y \longrightarrow M(Z), \ (f \colon A \longrightarrow Y) \mapsto \sum_{x \in A} \delta_{(x, f(x))}.$$

This induces the initial topology on $\mathcal{D}_Y$. By a slight abuse of language we call this topology the vague topology. Consider now the cover $\mathcal{U}$ of Z consisting of products of the form $B_{1/2} \times Y$ with $B_{1/2}$ an open ball with radius $1/2$ in X. Then, $\mathcal{D}_Y$ is a closed and hence compact subset of $M_{1,\mathcal{U}}(Z)$.

We can even consider a kind of universal Y as follows: Let $\mathbb{C}^*$ be an arbitrary compactification of $\mathbb{C}$. Then, $Y := \mathcal{D}_{\mathbb{C}^*}$ is a compact space by the preceeding considerations and so is then $\mathcal{D}_Y$. For an element $h \colon A \longrightarrow \mathbb{C}^*$ in Y, we set $\mathrm{dom}(h) := A$. It is not hard to see that

$$\tilde{\mathcal{D}} := \{ f \colon A \longrightarrow Y \mid \mathrm{dom}(f(x)) = A \text{ for all } x \in A \}$$

is a closed subset of the compact space $\mathcal{D}_Y$. Hence, $\tilde{\mathcal{D}}$ is compact as well. Any $f \colon A \longrightarrow Y$ satisfying $\mathrm{dom}(f(x)) = A$ gives rise to an $h \colon A \times A \longrightarrow \mathbb{C}^*$ with $h(x, y) := f(x)(y)$. We can and will therefore naturally identify $\tilde{\mathcal{D}}$ with the set

$$\{ (A, h) \mid A \in \mathcal{D}, h \colon A \times A \to \mathbb{C}^* \}.$$

For the use in the main text let us note that the topology of $\tilde{\mathcal{D}}$ is such that for any continuous $\phi \colon X \times X \times \mathbb{C}^* \to \mathbb{C}$ of compact support, the map

$$\tilde{\mathcal{D}} \to \mathbb{C}, \quad (A, h) \mapsto \sum_{x,y \in A} \phi(x, y, h(x, y))$$

is continuous. In particular, for a continuous $g \colon X \times X \to \mathbb{C}^*$ of compact support, the map $(A, h) \mapsto \sum_{x,y \in A} g(x, y) h(x, y)$ is continuous.

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